Matrix Equations (via example)

Problem: Given $\underline{A} = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix}$,

$\underline{b_1} = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$, $\underline{b_2} = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$, $\underline{b_3} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$, $\underline{b_4} = \begin{bmatrix} 6 \\ 5 \\ 1 \end{bmatrix}$,

(3x1)

Find vectors $\underline{x_1}$, $\underline{x_2}$, $\underline{x_3}$, and $\underline{x_4}$ such that

$$\underline{Ax_1} = \underline{b_1}, \quad \underline{Ax_2} = \underline{b_2}, \quad \underline{Ax_3} = \underline{b_3}, \quad \underline{Ax_4} = \underline{b_4}$$

Instead of solving 4 different problems/systems, set up like so:

- Finding 4 column vectors $\underline{x_i}$ is same as finding matrix $\underline{X} = [\underline{x_1}\ \underline{x_2}\ \underline{x_3}\ \underline{x_4}]_{(3\times 4)}$.
- 4 equations $\underline{Ax_1} = \underline{b_1}, \ldots, \underline{Ax_4} = \underline{b_4}$ is same as single equation:

$$[\underline{Ax_1}\ \ \underline{Ax_2}\ \ \underline{Ax_3}\ \ \underline{Ax_4}] = [\underbrace{\underline{b_1}\ \ \underline{b_2}\ \ \underline{b_3}\ \ \underline{b_4}}_{\underline{B}}]$$

$$\parallel$$

$$\underline{A}[\underline{x_1}\ \underline{x_2}\ \underline{x_3}\ \underline{x_4}] = \underline{B}$$

$$\parallel$$

$$\boxed{\underline{AX} = \underline{B}}$$

Assuming $\underline{A}^{-1}$ defined, then our desired $\underline{X} = [\underline{x_1}\ \underline{x_2}\ \underline{x_3}\ \underline{x_4}]$ is simply

$$\boxed{\underline{X} = \underline{A}^{-1}\underline{B}}$$

Example from book

Find a 3×4 matrix $\mathbf{X}$ such that

$$\overbrace{\begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix}}^{A} \mathbf{X} = \overset{\overbrace{[\,b_1 \quad b_2 \quad b_3 \quad b_4\,]}^{B}}{\begin{bmatrix} 3 & -1 & 2 & 6 \\ 7 & 4 & 1 & 5 \\ 5 & 2 & 4 & 1 \end{bmatrix}}.$$

The coefficient matrix is the matrix $\mathbf{A}$ whose inverse we found in Example 7, so Eq. (19) yields

$$\mathbf{X} = \underline{\mathbf{A}^{-1}}\mathbf{B} = \underline{\begin{bmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{bmatrix}} \begin{bmatrix} 3 & -1 & 2 & 6 \\ 7 & 4 & 1 & 5 \\ 5 & 2 & 4 & 1 \end{bmatrix},$$

and hence

$$\mathbf{X} = \begin{bmatrix} -4 & -13 & 14 & 1 \\ -1 & -5 & 8 & -2 \\ 11 & 33 & -39 & 4 \end{bmatrix}.$$

By looking at the third columns of $\mathbf{B}$ and $\mathbf{X}$, for instance, we see that the solution of

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 &= 2 \\ 5x_1 + 6x_2 + 3x_3 &= 1 \\ 3x_1 + 5x_2 + 2x_3 &= 4 \end{aligned}$$

is $x_1 = 14$, $x_2 = 8$, $x_3 = -39$. ■